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A Criterion for Toric Varieties

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A Criterion for Toric Varieties

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A Criterion for Toric Varieties

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We consider the pair of a smooth complex projective variety together with an anti-canonical simple normal crossing divisor (we call it "log Calabi-Yau"). Standard examples are toric varieties together with their toric boundaries (we call them "toric pairs"). We provide a numerical criterion for a general log Calabi-Yau to be toric by an inequality between its dimension, Picard number and the number of boundary components. The problem originates in birational geometry and our proof is constructive, motivated by mirror symmetry.

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Chapter 1

Statement of Main Theorem; Introduction

This thesis aim to analyze the rich geometry of Calabi-Yau objects using the idea of mirror symmetry. In particular, I will prove the following theorem:

Main Theorem 1. *Let $(Y, D = \sum_{1 \leq i \leq n} D_i)$ be a smooth complex projective variety with a simple normal crossing divisor (for the meaning of simple normal crossing, see Definition 2.1), satisfying $K_Y + D \equiv 0$. Let U be the group $\{(a_1, \ldots, a_n) \in \mathbb{Z}^n | \sum_{i=1}^{n} a_i D_i \equiv 0)\}$. Then we always have $\operatorname{rank}(U) \leq \dim(Y)$, and the equality holds if and only if the pair (Y, D) is a toric variety together with its toric boundary.*

This result was first conjectured by M$^{\text{c}}$Kernan, who also gave an unpublished sketch of a proof, assuming that the Minimal Model Program holds. My argument is based on entirely different ideas inspired by mirror symmetry, and more direct. The proof is presented in Chapters 2 and 3, divided into several steps.

Mirror symmetry (roughly and conjecturally) predicts that Calabi-Yau manifolds appear in pairs $(X, \check{X})$. My main interest is the SYZ mirror symmetry ([SYZ96]), which boils down to the following diagram:

$$\begin{tikzcd} X \arrow[rr, dashed, leftrightarrow] \arrow[d, "\phi"'] & & \check{X} \arrow[d, "\psi"] \\ B & B_0 \arrow[l, hook'] \arrow[r, hook] & \check{B}. \end{tikzcd}$$

Here, B_0 carries an integral affine structure, and is a common open subset of topological manifolds B and $\check{B}$ such that $\operatorname{codim}(B \setminus B_0) \geq 2$ and $\operatorname{codim}(\check{B} \setminus B_0) \geq 2$. The maps ϕ and ψ restrict to dual torus fibrations (SYZ fibrations) over B_0. Starting with a Calabi-Yau manifold X, we can follow this diagram to seek information about both X, and its "mirror" $\check{X}$. The explicit construction of $\check{X}$ is highly non-trivial. It requires delicate handling and has only been conquered for special cases ([KS06], [GS11a], [GMN09]).

Mirror symmetry can be extended to the setting of log Calabi-Yau varieties ([Au07]), i.e., instead of Y, we consider a pair (Y, D) of normal projective variety with a boundary, such that it has log canonical singularities (for its meaning, see Definition 5.1) and $K_Y + D \equiv 0$. The classical example is when Y is a smooth toric variety and D its toric boundary. In this case we have a well-behaved Landau-Ginzburg mirror model ([Ab06])–and a standard torus fibration from Y to a convex lattice polytope in $\mathbb{R}^d$, known as the moment map. The general 2-dimensional case, known as Looijenga pairs, was studied by Gross-Hacking-Keel ([GHK11]). Higher-dimensional cases are much subtler. However, thanks to a theorem of Kollár ([Kol11]), if (Y, D) is divisorial log terminal (for its meaning, see Definition 5.2) and $\mathbb{Q}$-factorial, then the dual simplicial complex Σ_D to D is pure and carries some strong connectivity, so at least has a nice topology. Natural questions then arise: can Σ_D be endowed with some reasonable integral affine structure? How far is the pair (Y, D) from being toric? And eventually, how does this help to study the mirror symmetry for (Y, D)?

This train of thought motivated my proof of the Main Theorem. Σ_D was endowed with certain linear functions such that when "the equality holds", it could be identified with a full-dimensional lattice polytope in $N_{\mathbb{R}}$. We then constructed a morphism from Y to the corresponding toric variety, and proved it was an isomorphism. This provides a torus fibration from Y to the polar set Σ_D°. Moreover, define the "charge" $0 \leq c(Y, D) = \dim(Y) + \operatorname{rank}(\operatorname{Pic}(Y)) - n$, which matches the number of non-toric blowups performed upon a toric pair. Then the theorem suggests that the charge somehow measures the "difficulty" to realize mirror symmetry for log Calabi-Yau varieties.

The Main Theorem also applies to the degenerations of Calabi-Yau varieties, which play central role in both mirror symmetry and compactifying moduli spaces. Log Calabi-Yau varieties appear as the "local models" in a degenerating family of Calabi-Yau varieties. More specifically, let $X \to \Delta \backslash \{0\}$ be a family of smooth Calabi-Yau varieties over a punctured disc. Suppose it has maximal monodromy, which roughly means the corresponding analytic arc in the moduli space of these varieties approaches a "maximal cusp", and is exactly the part of the moduli space where mirror symmetry is conjectured to exist ([Mo92]). One can complete it to a semi-stable model $\mathcal{X} \to \Delta$ by semi-stable reduction. Then the central fiber $\mathcal{X}^0$ is a K-trivial normal crossing union (for its meaning, see Definition 2.1) and all of its irreducible components are log Calabi-Yau varieties (Y, D)–where D is the intersection of Y with the singular locus of $\mathcal{X}^0$, or equivalently, the intersection of Y with other components of $\mathcal{X}^0$. The Clemens-Schmid spectral sequence implies that the dual complex B of the central fiber $\mathcal{X}^0$ is a triangulation of a manifold, and further a triangulation of a sphere when the generic fibers satisfy some restrictions on the Hodge numbers, which hold in particular when generic fibers are K3 surfaces ([FM83]).

Remarkable progress was made by Gross-Siebert ([GS11a]) towards realizing mirror symmetry under this setting. The mirror was explicitly constructed to a one-parameter toric degeneration of Calabi-Yau varieties–here "toric degeneration" means the central fiber is a (roughly normal crossing) union of toric varieties, glued along the toric boundaries in such a way that the dual complex is a polyhedral decompostion of a sphere. It then arises as a natural question to wonder how general this construction is, or, under what conditions one can apply the Gross-Siebert program. In particular, I would like to mention that Roberto Svaldi made an example of that a smooth log Calabi-Yau variety is not necessarily rational (but has be be rationally connected, by the following Proposition 8). So in philosophy, we should not consider Gross-Siebert program as the general situation.

The Main Theorem gives a weak solution to this question. It provides a criterion for recognizing toric pairs, and hence a "local" way of telling if the Gross-Siebert program will apply. However, a better candidate could be some conditions on the "global" structures of $\mathcal{X}$, and is something I am looking for.

A further wish would be to understand mirror symmetry according to the local charges.

First interesting examples of maximal degenerations are the type III semi-stable degenerations of K3 surfaces, of which every component (Y, D) of the central fibers is a Looijenga pair. Under certain (different) assumptions, we have multiple ways to define integral affine structures with singularities on B, which is a triangulation of S^2:

1. (Gross-Siebert, [GS11b]) The degeneration induces a log structure on $\mathcal{X}^0$, of which we can take the tropicalization.

2. (Kontsevich-Soibelman, [KS06]) The generic fiber X can be viewed as a variety over a non-archimedean field, of which we can take the corresponding Berkovich space X^{an}. The specialization map then induces a projection $X^{an} \to B$, which looks locally like the standard projection $(\mathbb{T}^2)^{an} \to \mathbb{R}^2$ and determines an integral affine structure with singularities on B. A main feature of this construction is that it is independent up to retraction maps of the different choices of the central fibers filling in to get models with simple normal crossings, by the weak factorization ([AKMW02]).

3. (Gross-Hacking-Keel, [GHK11]) This is a combinatorial construction, and the most explicit one. One makes an affine structure on B by defining the affine functions as those piecewise affine function satisfying certain conditions. These conditions rely mainly on the self-intersection numbers of the irreducible components of D, considered as rational curves in Y.

Each of these structures captures the information of a degenerating family in different aspects. In particular, it is conjectured ([KS06]) that the Kontsevich-Soibelman construction provides a basis of an SYZ fibration. A natural speculation, inspired by mirror symmetry (canonicality), is that they are isomorphic (or maybe polar sets of one another), up to some controllable operations. This question serves as a natural subsequent problem and is expected to be solved in the future.

Examples of the Main Theorem are presented in Chapter 4, and more backgrounds/remarks are included in Chapter 5.

Chapter 2

Proof of Inequality

Definition 2.1. Let k be a field, Y a k-scheme and $D = \sum a_i D_i$ a Weil divisor on Y with the D_i irreducible. We say that (Y, D) has snc (= simple normal crossing) at a point $p \in Y$ if Y is smooth at p and there is an open neighborhood $p \in Y_p \subseteq Y$ and coordinates $y_1, \ldots, y_d$ on Y_p such that $Y_p \cap \operatorname{Supp} D \subseteq (y_1 \cdots y_d = 0)$. We say that (Y, D) is snc if it is snc at every point. Note that being simple normal crossing is local in the Zariski topology, but not in the étale topology. Given (Y, D), the largest open set $U \subseteq Y$ such that $(U, D|_U)$ is snc is called the snc locus of (Y, D), denoted by $\operatorname{snc}(Y, D)$.

We say that (Y, D) has nc (= normal crossing) at a point $p \in Y$ if there is an étale neighborhood $\pi : (p' \in Y') \to (p \in Y)$ such that $(Y', \pi^{-1} D)$ is snc at p'. We say that (Y, D) is nc if it is nc at every point. Being normal crossing is local in the étale topology.

Example 2.1. *Let $p \in D$ be a nc point of multiplicity 2. If $\operatorname{char}(k) \neq 2$, then in suitable local coordinates, D can be given by an equation $x_1^2 - u x_2^2 = 0$ where $u \in \mathcal{O}_{p,Y}$ is a unit. D is snc at p iff u is a square in $\mathcal{O}_{p,Y}$.*

For example, $(y^2 - (1 + x)x^2 = 0) \subset \mathbb{A}^2_k$ is nc but it is not snc at the origin. Similarly, $(x^2 + y^2 = 0) \subset \mathbb{A}^2_k$ is nc but it is snc only if $\sqrt{-1} \in k$.

Now let $(Y, D = \sum_{1\leq i\leq n} D_i)$ be a smooth complex projective variety with a simple normal crossing divisor, satisfying $K_Y + D = 0$. Denote $d = \dim(Y)$, $U = \ker(\mathbb{Z}^D \to \mathrm{Pic}(Y))$. For $I \subseteq \{1, \dots, n\}$, set $D_I = \cap_{i\in I} D_i$ and call it a stratum in D (in particular, $D_\emptyset = Y$).

Lemma 1. *The Main Theorem holds if and only if the same holds under the additional assumption that each stratum is smooth.*

Proof. Suppose there exists a singular stratum. We can make the resolution of singularities and get a new pair $(\tilde{Y}, \tilde{D})$, where $\tilde{D}$ is the reduced inverse image of D. Then $(\tilde{Y}, \tilde{D})$ satisfies the same condition with (Y, D) above, and determines the same group U. Moreover, $\tilde{Y} \setminus \tilde{D} = Y \setminus D$, and $(\tilde{Y}, \tilde{D})$ is a toric variety together with its toric boundary if and only if the same holds for (Y, D) (because every exceptional divisor is an orbit closure under the action of torus). $\square$

From now on we will assume that for any $I \subseteq \{1, \dots, n\}$, the stratum D_I is smooth. As the first step, the purpose of this chapter is to prove

Theorem 2. $\mathrm{rank}(U) \leq d$.

Proposition 2. *There exists a canonical identification $U \simeq \mathcal{O}^\times(Y \setminus D)/\mathbb{C}^\times$, associating each $u \in U$ to some rational function r_u on Y that is regular and non-vanishing on $Y \setminus D$, satisfying $\mathrm{val}_{D_i}(r_u) = u(D_i)$ for each $1 \leq i \leq n$.*

Proof. This follows from the exact sequence

$$0 \to \mathbb{C}^\times \to \mathcal{O}^\times(Y \setminus D) \xrightarrow{r \mapsto \sum_i \mathrm{val}_{D_i}(r) D_i} \mathbb{Z}^D \to \mathrm{Pic}(Y).$$

$\square$

Lemma 3. *For any $1 \le i \le n$, set $E_i = \sum_{j \neq i}(D_j \cdot D_i)$, then (D_i, E_i) is a smooth projective variety with a simple normal crossing divisor, and $K_{D_i} + E_i = 0$. Moreover, for any $I \subseteq \{1, \ldots, n\}$, set $E_I = \sum_{j \notin I}(D_j \cdot D_I)$, then (D_I, E_I) is a smooth projective variety with a simple normal crossing divisor, and $K_{D_I} + E_I = 0$.*

Proof. By adjunction formula, $K_{D_i} = (K_Y + D_i)|_{D_i}$, so $K_{D_i} + E_i = (K_Y + D)|_{D_i} = 0$. The second statement follows by iteratively using the first. $\square$

The following definition and proposition are adopted from [Kol11].

Definition 2.2 ([Kol11], Definition 9)**.** A divisorial log terminal, $\mathbb{Q}$-factorial pair $(X, Z_1 + Z_2)$ with Z_1, Z_2 the sole log canonical centers (for its meaning, see pp. 31) in X is called a standard $\mathbb{P}^1$-link if there exists a proper morphism $\pi : X \to S$ such that $K_X + Z_1 + Z_2 \sim_{\mathbb{Q},\pi} 0$, $\pi : Z_i \to S$ are both isomorphisms and every reduced fiber is isomorphic to $\mathbb{P}^1$.

Proposition 4 ([Kol11], Theorem 10)**.** *All minimal strata in D are birationally equivalent. Moreover, for any two $D_I, D_{I'}$ of them, there is a sequence of minimal strata $D_I = D_{I_1}, D_{I_2}, \ldots, D_{I_k} = D_{I'}$ such that for each $1 \le i \le k-1$, there is a stratum W_i containing D_{I_i} and $D_{I_{i+1}}$, and the pair $(W_i, D_{I_i} + D_{I_{i+1}})$ is birational to a standard $\mathbb{P}^1$-link with D_{I_i} mapping to Z_1 and $D_{I_{i+1}}$ mapping to Z_2.*

We can construct the dual simplicial complex Σ_D to $D = \sum_{1\le i\le n} D_i$: the 0-simplices are labeled by the irreducible components of D and for every stratum $D_I \neq \emptyset$ we attach a $(|I|-1)$-dimensional cell. By Proposition 4, there are two cases: D is disconnected, then Σ_D is simply two points; D is connected, then

Proposition 5. *Σ_D is a pure simplicial complex. I.e., all largest simplices in Σ_D have the same dimension $0 < l < d$, and every simplex of dimension less than l is a face of some simplex $\sigma \in \Sigma_D^l$, of dimension exactly l. Moreover, any two l-simplices $\sigma, \sigma' \in \Sigma_D^l$ can be connected by a chain of l-simplices $\sigma_1 = \sigma, \sigma_2, \ldots, \sigma_k = \sigma'$ such that for each $1 \le i \le k-1$, σ_i and σ_{i+1} share a $(l-1)$-dimensional face $\tau_i \in \Sigma_D^{l-1}$, and τ_i is the face of exactly two l-simplices.*

Definition 2.3. Let $f : \Sigma_D^0 \to \mathbb{Z}$ be an integer-valued function on the set of vertices in Σ_D. Consider a pair $\sigma_1, \sigma_2 \in \Sigma_D^l$ sharing a face $\tau \in \Sigma_D^{l-1}$, which corresponds to two minimal strata D_{I_1}, D_{I_2} (whose dimension is $d-l-1$) contained in a $(d-l)$-dimensional stratum W such that the pair $(W, D_{I_1} + D_{I_2})$ is birational to a standard $\mathbb{P}^1$-link. Then f is called linear across τ if

$$\Big(\sum_{P \in (\sigma_1 \cup \sigma_2)^0} f(P) D_P \Big) \cdot C = 0,$$

where D_P is the component of D corresponding to P, C is a generic $\mathbb{P}^1$-fiber of $(W, D_{I_1} + D_{I_2})$. f is called linear if for every pair $\sigma_1 \cap \sigma_2 = \tau$ as above, it is linear across τ. Denote by $L(\Sigma_D)$ the group of linear functions on Σ_D.

The degenerate case is when D is disconnected, Σ_D is the union of two points, and a linear function is defined as taking opposite values on them.

Proposition 6. *(1) There exists a natural embedding $U \hookrightarrow L(\Sigma_D)$, associating each $u \in U$ to a linear function $f_u : \Sigma_D^0 \to \mathbb{Z}$.*

(2) For any $\sigma \in \Sigma_D^l$, the restriction map $L(\Sigma_D) \to \mathbb{Z}^{\sigma^0}$ is injective.

Proof. (1) There is a natural identification $\mathbb{Z}^D \simeq \mathbb{Z}^{\Sigma_D^0}$. Then it suffices to show f_u is linear across each $\tau = \sigma_1 \cap \sigma_2$, which is equivalent to $(\sum_i u(D_i)D_i).C = 0$, and follows directly from $\sum_i u(D_i)D_i = 0$ in $\mathrm{Pic}(Y)$.

(2) A linear function that vanishes on σ^0 must vanish everywhere, by Proposition 4. □

Corollary 7. *U is a free abelian group with $\mathrm{rank}(U) \leq l+1 \leq d$. In particular, if $\mathrm{rank}(U) = d$, then $l+1 = d$, which implies all minimal strata in D are points.*

Proof. By Proposition 6, $U \hookrightarrow L(\Sigma_D) \hookrightarrow \mathbb{Z}^{\sigma^0}$, which is isomorphic to the free abelian group $\mathbb{Z}^{l+1}$. □

This finishes the proof of Theorem 2. The following is another nice property of such a pair (Y, D), which we will use later.

Proposition 8. *(1) Y is rationally connected. Moreover, any stratum that is not minimal is rationally connected.*

(2) When $d = 2$, Y is a rational surface and every $D_i, 1 \leq i \leq n$ is a rational curve.

Proof. (1) Apply induction on d. For $d = 1$ we have Y is Fano and hence rational. For $d > 1$, by Lemma 3, every component D_i of D satisfies the assumption and so, by induction hypothesis, is rationally connected.

Suppose Y is not rationally connected. Consider the maximal rationally connected fibration $\varphi : Y \dashrightarrow Z$ with $0 < \dim(Z) < d$. Take a (general) rationally connected fiber Y_z, $z \in Z$. Then $\dim(Y_z) = \dim(Y) - \dim(Z) > 0$. We claim that D has a non-empty intersection with Y_z. This is trivial if $\dim(Y_z) > 1$. Suppose $\dim(Y_z) = 1$, then the adjunction formula gives $D \cdot Y_z = -K_Y \cdot Y_z = -K_{Y_z}$. Since Y_z is rationally connected, K_{Y_z} is not trivial, which implies our claim.

In this way we get a dominant rational map $D \dashrightarrow Z$. Then Z is rationally connected since each D_i is. This happens only if Z is a single point, and produces a contradiction. Hence Y is rationally connected. The second statement then follows from Lemma 3.

(2) For curves and surfaces, being rationally connected is equivalent to being rational. □

Corollary 9. *Σ_D is a homology sphere, i.e., $H^r(\Sigma_D, \mathbb{C}) = 0$ unless $r = 0$ or $r = d - 1$, and $\dim H^0(\Sigma_D, \mathbb{C}) = \dim H^{d-1}(\Sigma_D, \mathbb{C}) = 1$.*

Proof. By Proposition 8, every stratum D_I with $\dim(D_I) > 0$ is rationally connected, so $H^r(D_I, \mathcal{O}_{D_I}) = 0$ for every $r > 0$ and $I \subseteq \{1, \ldots, n\}$. We can then use [FM83], pp.26-27 and get $H^r(D, \mathcal{O}_D) = H^r(\Sigma_D, \mathbb{C})$ for every r.

Consider the exact sequence

$$0 \to \mathcal{O}(-D) \to \mathcal{O}_Y \to \mathcal{O}_D \to 0,$$

and notice that $H^i(Y, \mathcal{O}(-D)) = H^i(Y, K_Y) \approx H^{d-i}(Y, \mathcal{O}_Y)$. The induced long exact sequence gives $H^r(D, \mathcal{O}_D) = 0$ unless $r = 0$ or $r = d-1$, and $\dim H^0(D, \mathcal{O}_D) = \dim H^{d-1}(D, \mathcal{O}_D) = 1$. $\square$

Chapter 3

Proof of Equivalence Condition

3.1 Proof of Sufficiency

The sufficiency follows immediately from the following standard property of toric varieties:

Theorem 3 ([Fu93], pp. 63). *Let $Y = Y(\Delta)$ be the toric variety corresponding to a fan Δ not contained in any proper subspace of $N_{\mathbb{R}}$. Then there is a commutative diagram with exact rows:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & \mathrm{Div}_T(Y) & \longrightarrow & \mathrm{Pic}(Y) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} & & \\
0 & \longrightarrow & M & \longrightarrow & \oplus_{i=1}^{n}\mathbb{Z}\cdot D_i & \longrightarrow & A_{n-1}(Y) & \longrightarrow & 0,
\end{array}$$

where i is the natural embedding, $\mathrm{A}_{n-1}(Y)$ is the group of all Weil divisors modulo the subgroup of divisors $[\mathrm{div}(f)]$ of rational functions, and j is the embedding determined by $D \mapsto [D]$.

In our case Y is smooth projective, so $\mathrm{Div}_{\mathrm{T}}(Y) = \oplus_{i=1}^{n}\mathbb{Z}\cdot D_i$, and $\mathrm{rank}(U) = \mathrm{rank}(M) = d$.

3.2 Proof of Case $d = 2$

In this section we will prove the Main Theorem for the case Y is a surface (hence rational, by Proposition 8). We need the following definition and proposition adopted from [GHK11]:

Definition 3.1 ([GHK11], Definition 1.16-1.18)**.** Let Y be a rational surface and D an anti-canonical cycle of rational curves on Y. A toric blow-up of the pair (Y, D) is a birational morphism $\pi : \tilde{Y} \to Y$ such that if $\tilde{D}$ is the reduced scheme structure on $\pi^{-1}(D)$, then $\tilde{D}$ is an anti-canonical cycle of rational curves on $\tilde{Y}$. Equivalently, a toric blow-up is a blow-up of Y along a subscheme supported on $\mathrm{Sing}(D)$. A toric model of (Y, D) is a birational morphism $(Y, D) \to (\bar{Y}, \bar{D})$ to a smooth toric surface with its toric boundary such that $D \to \bar{D}$ is an isomorphism.

Proposition 10 ([GHK11], Proposition 1.19)**.** *Given (Y, D) there exists a toric blowup $(\tilde{Y}, \tilde{D})$ which has a toric model $(\tilde{Y}, \tilde{D}) \to (\bar{Y}, \bar{D})$.*

Similar to the group U, we can define $\tilde{U}$ and $\bar{U}$ corresponding to the pairs $(\tilde{Y}, \tilde{D})$ and $(\bar{Y}, \bar{D})$ respectively.

Lemma 11. *For any toric blow-up $(\tilde{Y}, \tilde{D})$ of (Y, D), $\tilde{U} \simeq U$.*

Proof. Without loss of generality, let $\pi : \tilde{Y} \to Y$ be the toric blow-up along a single stratum D_I, $I \subseteq \{1, \dots, n\}$ and $|I| \geq 2$ (since $D_I \subseteq \mathrm{Sing}(D)$). Denote the exceptional divisor by $\tilde{D}_I$. π determines two canonical isomorphisms $\mathbb{Z}^{\tilde{D}} \simeq$

$\mathbb{Z}^D \oplus \mathbb{Z}^{\tilde{D}_I}$ and $\mathrm{Pic}(\tilde{Y}) \simeq \mathrm{Pic}(Y) \oplus \mathbb{Z}^{\tilde{D}_I}$, hence a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}^{\tilde{D}} & \longrightarrow & \mathbb{Z}^{D} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathrm{Pic}(\tilde{Y}) & \longrightarrow & \mathrm{Pic}(Y) & \longrightarrow & 0. \end{array}$$

Using Snake Lemma, we get $\tilde{U} \simeq U$. □

Theorem 4. *The Main Theorem holds when $d = 2$.*

Proof. Since $(\bar{Y}, \bar{D})$ is a toric pair, $\mathrm{rank}(\bar{U}) = d$. Consider the morphism $p : (\tilde{Y}, \tilde{D}) \to (\bar{Y}, \bar{D})$. Denote the exceptional locus by by $\tilde{E}$. π determines two canonical morphisms $\mathbb{Z}^{\tilde{D}} \simeq \mathbb{Z}^D$ and $\mathrm{Pic}(\tilde{Y}) \simeq \mathrm{Pic}(Y) \oplus \mathbb{Z}^{\tilde{E}}$. It is then easy to see (or by a similar argument to Lemma 11) that $\mathrm{rank}(\tilde{U}) < \mathrm{rank}(\bar{U}) = d$, and the equality holds if and only if $\tilde{Y} = \bar{Y}$. The statement then follows from Lemma 11. □

3.3 Construction of the Fan

From now on we return to the case of general dimension. We assume $\mathrm{rank}(U) = d$ and, unless otherwise specified, $d > 1$.

Proposition 12. *Let C be an irreducible 1-stratum in Y, corresponding to a $(d-2)$-simplex $\sigma_C \in \Sigma_D^{d-2}$. Then $C \simeq \mathbb{P}^1$ and it contains exactly two 0-strata, or equivalently, there exist exactly two $(d-1)$-simplices $\sigma_1, \sigma_2 \in \Sigma_D^{d-1}$ containing σ_C as a face.*

Proof. Denote by $\{p_1, \ldots, p_m\}$ the set of 0-strata on C, then $K_C + \sum_{1 \le i \le m} p_i = 0$ by Lemma 3. Now $m > 0$, so C is Fano, hence rational. This also follows directly from Proposition 8. □

For any $1 \le i \le n$, set $E_i = \sum_{j \neq i} (D_j \cdot D_i)$, we define two groups $U^i = \{u \in U = \ker(\mathbb{Z}^D \to \mathrm{Pic}(Y)) | u(D_i) = 0\}$ and $U_i = \ker(\mathbb{Z}^{E_i} \to \mathrm{Pic}(D_i))$. Consider the group $N := \mathrm{Hom}(U, \mathbb{Z})$. It is free abelian of $\mathrm{rank}(N) = \mathrm{rank}(U) = d$ and can be viewed as a lattice in $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$. Similarly, $N_i := \mathrm{Hom}(U_i, \mathbb{Z})$ can be viewed as a lattice in $(N_i)_\mathbb{R} = N_i \otimes_\mathbb{Z} \mathbb{R}$. Moreover, denote the vertex in Σ_D corresponding to D_i by P_i, we define two simplicial complexes: Σ^i is the union in Σ_D of all $(d-1)$-simplices containing P_i, Σ_i is the dual simplicial complex to E_i as a divisor of D_i.

Lemma 13. *(1) Σ^i is a cone over Σ_i. In particular, there is a natural identification of simplices con* $: \Sigma_i^{d-2} \xrightarrow{\sim} (\Sigma^i)^{d-1} : \sigma \mapsto \mathrm{Conv}(\sigma, P_i)$.

(2) There is a natural monomorphism of groups res $: U^i \hookrightarrow U_i$, *with the same rank* $= (d-1)$.

(3) There is a monomorphism of groups $U \hookrightarrow U_i \oplus \mathbb{Z}$, *with the same rank* $= d$. *Hence,* $N_\mathbb{R} \simeq (N_i)_\mathbb{R} \oplus \mathbb{R}$.

Proof. (1) Easy.

(2) Pick an element $u \in U^i$. The corresponding meromorphic function r_u has neither zeros nor poles on $Y \setminus D$, and D_i is not contained in the support of principal divisor (r_u). It would follow that r_u has neither zeros nor poles on

$D_i \setminus E_i$ (otherwise the support of (r_u) must intersect $Y \setminus D$), thus lies in U_i. This defines a group homomorphism $res : U^i \to U_i$.

Suppose $res(u) = 0$. Then r_u has neither zeros nor poles on $Y \setminus D$, D_i is not contained in the support of (r_u), and no component of E_i is contained in the support of (r_u). So any component D_j of D with with $D_j \cap D_i \neq \emptyset$ is not contained in the support of (r_u), hence $u(D_j) = 0$. In particular, pick a 0-stratum p on D_i with corresponding simplex $\sigma_p \in \Sigma_D^{d-1}$, then u vanishes on σ_p^0, so must vanish everywhere, i.e., $u = 0$. Hence res is injective.

By definition, $\mathrm{rank}(U^i) \geq \mathrm{rank}(U) - 1 = d - 1$; by Corollary 7, $\mathrm{rank}(U_i) \leq \dim(D_i) = d - 1$. So $\mathrm{rank}(U^i) = \mathrm{rank}(U_i) = d - 1$.

(3) Pick an element $u_0 \in U \setminus U^i$, i.e., $u_0(D_i) \neq 0$. Then we have a homomorphism $U \to U^i \oplus \mathbb{Z} : u \mapsto (u_0(D_i)u - u(D_i)u_0, u(D_i))$, which is obviously a monomorphism. Its composition with $(res, 1)$ gives the desired.

□

Define a map $\psi : \Sigma_D^0 \to N : P \mapsto (n_P : u \mapsto f_u(P))$. It can be extended to a continuous piecewise linear map $\psi : \Sigma_D \to N_{\mathbb{R}}$ in an obvious way. Set $\Sigma = (\Sigma_D \times \mathbb{R}_{\geq 0})/(\Sigma_D \times \{0\})$ to be the unbounded cone over Σ_D and call it the dual simplicial complex to the pair (Y, D). Then ψ can be further extended by linearity to a continuous map $\Psi : \Sigma \to N_{\mathbb{R}}$, mapping the vertex $v = [\Sigma_D \times \{0\}]$ to 0 and $(P, 1)$ to n_P for each $P \in \Sigma_D^0$. By the following Lemma 14, $\Psi^{-1}(0) = v$, so we may restrict to $\bar{\Psi} = \Psi|_{\Sigma \setminus v} : \Sigma \setminus v \to N_{\mathbb{R}} \setminus 0$.

Lemma 14. *Let $\sigma_1, \sigma_2 \in \Sigma_D^{d-1}$ be two largest simplices sharing a face $\tau \in \Sigma_D^{d-2}$. Then ψ is injective on $\sigma_1 \cup \sigma_2$.*

Proof. Firstly we prove ψ is injective on each simplex. It suffices to show that, for any $\sigma \in \Sigma_D^{d-1}$ with $\sigma^0 = \{P_1, \dots, P_d\}$, the images $\{\psi(P_1), \dots, \psi(P_d)\}$ are linearly independent in $N_{\mathbb{R}}$. Suppose to the contrary that there exists $(a_1, \dots, a_d) \neq 0$ with $\sum_{i=1}^d a_i \psi(P_i) = 0$. Then $\sum_{i=1}^d a_i n_{P_i} = 0$, i.e., $\sum_{i=1}^d a_i f_u(P_i) = 0$ for any $u \in U$. This implies $U \subseteq \{u \in \mathbb{Z}^{\sigma^0} \mid \sum_{i=1}^d a_i f_u(P_i) = 0\}$. The right-hand side is a free abelian group of rank $d-1$, a contradiction to $\operatorname{rank}(U) = d$.

Now let $\sigma_1^0 = \{Q_1, \dots, Q_d\}$ and $\sigma_2^0 = \{Q_2, \dots, Q_{d+1}\}$. By Proposition 6, U is full-rank subgroup of $\mathbb{Z}^{\sigma_1^0}$, so there exists $u \in U$ with $f_u(Q_1) > 0$ and $f_u(Q_i) = 0$ for $2 \leq i \leq d$. Then by Definition 2.3, $(f_u(Q_1)D_1 + f_u(Q_{d+1})D_{d+1}) \cdot C = 0$, where D_1, D_{d+1}, C are the corresponding strata to Q_1, Q_{d+1}, τ respectively. Proposition 12 implies $D_1 \cdot C = D_{d+1} \cdot C = 1$, and in particular $f_u(Q_{d+1}) < 0$. So the function f_u separates σ_1 and σ_2 by different signs. Hence $\psi(\operatorname{int}(\sigma_1)) \cap \psi(\operatorname{int}(\sigma_2)) = \emptyset$. □

Lemma 15. *$\bar{\Psi}$ is proper.*

Proof. *Claim 1: $\bar{\Psi}$ is a closed map.*

Denote $\Sigma_D^{d-1} = \{\sigma_1, \dots, \sigma_m\}$, $R(\sigma_i) = \sigma_i \times \mathbb{R}_{>0}$ for $1 \leq i \leq m$. Let A be a closed subset of $\Sigma \setminus v$, $A_i := A \cap R(\sigma_i)$. Then $A = \cup_{i=1}^m A_i$, which is a finite union of closed subsets. By Lemma 14, $\bar{\Psi}$ is a homeomorphism on each $R(\sigma_i)$, so $\bar{\Psi}(A_i)$ is closed in $N_{\mathbb{R}} \setminus 0$. Then $\bar{\Psi}(A) = \cup_{i=1}^m \bar{\Psi}(A_i)$ is also closed.

Claim 2: for any $x \in N_{\mathbb{R}} \setminus 0$, $\bar{\Psi}^{-1}(x)$ is finite.

$\bar{\Psi}$ is a homeomorphism on each $R(\sigma_i)$, so $|\bar{\Psi}^{-1}(x) \cap R(\sigma_i)| \leq 1$ for each $1 \leq i \leq m$. So $|\bar{\Psi}^{-1}(x)| = |\cup_{i=1}^{m} (\bar{\Psi}^{-1}(x) \cap R(\sigma_i))| \leq m$. □

Theorem 5. *Ψ is a homeomorphism, and identifies Σ with a complete fan in $N_{\mathbb{R}}$.*

Proof. Apply induction on d.

Step 1: When $d = 2$, $\bar{\Psi}$ is a covering map, and Ψ is a homeomorphism.

This follows from Theorem 4. But in order to make our idea more clear, let me make a different argument of the first statement. By Lemma 15, $\bar{\Psi}$ is proper, so it suffices to show it is a local homeomorphism. Define open subset $R(\sigma_1, \sigma_2) = \mathrm{int}(\sigma_1 \cup \sigma_2) \times \mathbb{R}_{>0}$, where $\sigma_1, \sigma_2 \in \Sigma_D^{d-1}$ are two largest simplices (=rays) sharing a face (=vertex) $\tau \in \Sigma_D^{d-2}$. By Proposition 12, this determines a finite open covering of Σ. By Lemma 14, $\bar{\Psi}$ is a homeomorphism on each of them.

Step 2: When $d > 2$, $\bar{\Psi}$ is a covering map.

As in *Step 1*, we will show it is a local homeomorphism. Fix a point $P_i \in \Sigma_D^0$, corresponding to boundary component D_i. Write $(\Sigma^i)^0 = \{Q_1, \ldots, Q_m, P_i\}$. Then $(\Sigma_i)^0 = \{Q_1, \ldots, Q_m\}$, and each Q_j $(1 \leq j \leq m)$ corresponds to some $D_{i_j} \cap D_i \neq \emptyset$. Denote $R(\Sigma^i) = \mathrm{int}(\Sigma^i) \times \mathbb{R}_{>0}$, then $\{R(\Sigma^i) | 1 \leq i \leq n\}$ consists a finite open covering of $\Sigma \setminus v$. By Lemma 13, the pair (D_i, E_i) satisfies the same assumption as (Y, D). So by induction hypothesis, we get a homeomorphism Ψ_i from $(\Sigma_i \times \mathbb{R}_{\geq 0})/(\Sigma_i \times \{0\})$ to $(N_i)_{\mathbb{R}}$. Then by the following Lemma 16, $\bar{\Psi}$ is a homeomorphism on Σ^i, and hence a homeomorphism on $R(\Sigma^i)$.

Step 3: When $d > 2$, Ψ is a homeomorphism.

$\bar{\Psi}$ must be a homeomorphism, by *Step 2* and $\pi_1(N_{\mathbb{R}} \setminus 0) = 0$. Then so is Ψ. $\square$

Lemma 16. *Under the isomorphism $N_{\mathbb{R}} \simeq (N_i)_{\mathbb{R}} \oplus \mathbb{R}$ settled in Lemma 13(3),*

$$\Psi((P,1)) = \begin{cases} \left(\frac{\Psi_i((P,1))}{u_0(D_i)}, \frac{f_{u_0}(P)}{u_0(D_i)}\right), & P \in \{Q_1, \dots, Q_m\} \\ (0,1), & P = P_i. \end{cases}$$

Proof. Let $P \in \{Q_1, \dots, Q_m\}$, then for each $u \in U$, $\left(\frac{\Psi_i((P,1))}{u_0(D_i)}, \frac{f_{u_0}(P)}{u_0(D_i)}\right)(u) = \frac{\Psi_i((P,1))}{u_0(D_i)}(u_0(D_i)u - u(D_i)u_0) + \frac{f_{u_0}(P)}{u_0(D_i)}u(D_i) = \frac{f_{(u_0(D_i)u - u(D_i)u_0)}(P)}{u_0(D_i)} + \frac{f_{u_0}(P)}{u_0(D_i)}u(D_i) = \frac{u_0(D_i)f_u(P) - u(D_i)f_{u_0}(P)}{u_0(D_i)} + \frac{f_{u_0}(P)}{u_0(D_i)}u(D_i) = f_u(P) = \Psi((P,1))(u)$.

Let $P = P_i$, then for each $u \in U$, $(0,1)(u) = u(D_i) = \Psi((P_i,1))(u)$. $\square$

3.4 Proof of Necessity

Pick a point $e \in Y \setminus D$. Let p be a 0-stratum in D, corresponding to a largest simplex $\sigma_p \in \Sigma_D^{d-1}$ with $\sigma_p^0 = \{P_1, \dots, P_d\}$. Denote by U^p the complement in Y of all components of D not containing p. Define a rational map $\phi_p : U^p \dashrightarrow \mathrm{TV}(R(\sigma_p)) = \mathrm{Spec}(\mathbb{C}[\check{\sigma}_p \cap U])$ corresponding to the ring homomorphism $\mathbb{C}[\check{\sigma}_p \cap U] \to K(U^p) : \chi^u \mapsto (y \mapsto r_u(y)/r_u(e))$ (this is independent of the choice of r_u). Then the maps $\{\phi_p | p \text{ is a 0-stratum in } D\}$ naturally glue and become a rational map $\phi : Y \dashrightarrow X := \mathrm{TV}(\Sigma)$.

Proposition 17. *(1) ϕ is a regular morphism.*

(2) For each $1 \le i \le n$, let $P_i \in \Sigma^0$ be the vertex corresponding to boundary component D_i, X_i be the toric boundary of X corresponding to the ray $R(P_i) = P_i \times \mathbb{R}_{\ge 0}$, then $\phi^{-1}(X_i) = D_i$.

Proof. (1) For each 0-stratum p, Let D_i be a component of D containing p, then it corresponds to some $P_i \in \sigma_p^0$, $1 \le i \le d$. We have $\mathrm{val}_{D_i}(r_u) = u(D_i) = f_u(P_i)$. It must be non-negative since $R(P_i)$ is a ray of σ_p, while $u \in \check{\sigma}_p \cap U$. So $r_u \in \mathcal{O}(U^p)$, and hence ϕ_p is regular. This implies ϕ is regular.

(2) $X_i = \mathrm{Spec}(\mathbb{C}[\check{\sigma}_p \cap P_i^{\perp} \cap U])$, so a function $\chi^u \in \mathcal{O}(\mathrm{TV}(R(\sigma_p))) = \mathbb{C}[\check{\sigma}_p \cap U]$ vanishes on X_i if and only if $f_u(P_i) > 0$, i.e., $u(D_i) > 0$, i.e., r_u vanishes on D_i. Pick an affine subset $\mathrm{Spec}(S) \subseteq U^p$, denote the ring homomorphism $\rho : \mathbb{C}[\check{\sigma}_p \cap U] \to S$ and the associated morphism $\rho^a : \mathrm{Spec}(S) \to \mathrm{Spec}(\mathbb{C}[\check{\sigma}_p \cap U])$. Then $(\rho^a)^{-1}(X_i) = (\rho^a)^{-1}(V(I(X_i))) = V(\rho(I(X_i))) \supseteq D_i \cap \mathrm{Spec}(S)$. This implies $\phi(D_i) \subseteq X_i$.

Consider the morphism $\phi' : Y \setminus D \to \mathrm{Spec}(\mathbb{C}[U])$ corresponding to the ring homomorphism $\mathbb{C}[U] \to \mathcal{O}(Y \setminus D) : \chi^u \mapsto (y \mapsto r_u(y)/r_u(e))$. For any 0-stratum p, its composition with the canonical open immersion $\mathrm{Spec}(\mathbb{C}[U]) \hookrightarrow \mathrm{Spec}(\mathbb{C}[\check{\sigma}_p \cap U])$ would agree with $\phi_p|_{Y \setminus D}$ since they correspond to the same homomorphism $\mathbb{C}[\check{\sigma}_p \cap U] \to \mathcal{O}(Y \setminus D)$. In particular, $\phi|_{Y \setminus D} = \phi'$, and so $\phi(Y \setminus D) \subseteq \mathrm{Spec}(\mathbb{C}[U]) = X \setminus (\cup_i X_i)$. This, together with above, implies $\phi^{-1}(X_i) = D_i$. $\square$

Theorem 6. *(1) For each $1 \le i \le n$, D_i is a toric variety, and ϕ restricts to a finite toric morphism $\phi^i : D_i \to X_i$.*

(2) Y is a toric variety, and $\phi : Y \to X$ is a finite toric morphism.

Proof. Apply induction on d.

Step 1: (2) holds for $d = 1$.

By Proposition 12, $Y \simeq \mathbb{P}^1$ and D is the sum of two points on Y. So Σ is a line and $\mathrm{TV}(\Sigma) = \mathbb{P}^1$. ϕ is an isomorphism mapping D to the toric boundary $\{0\} + \{\infty\}$.

Step 2: (2) holds for $d - 1$ implies (1) holds for d.

By Lemma 13, the pair (D_i, E_i) satisfies the same assumption as (Y, D). Let $R(\Sigma_i) = (\Sigma_i \times \mathbb{R}_{\ge 0})/(\Sigma_i \times \{0\})$. By (2), D_i is a toric variety, and we have a finite toric morphism $\phi_i : D_i \to \mathrm{TV}(R(\Sigma_i))$. On the other hand, let $S(\Sigma^i)$ be the star of $P_i \times \mathbb{R}_{\ge 0}$ in Σ, then $X_i \simeq \mathrm{TV}(S(\Sigma^i))$. The embedding $U^i \hookrightarrow U_i$ settled in Lemma 13(2) induces a finite toric morphism $\mathrm{TV}(R(\Sigma_i)) \to \mathrm{TV}(S(\Sigma^i))$, and hence a finite toric morphism $\phi^i : D_i \to X_i$.

Step 3: (1) holds for d implies (2) holds for d.

By Proposition 17, the finiteness of ϕ_i implies ϕ has a finite degree on D_i, and hence is finite by the upper semi-continuity.

Let R be the ramification divisor of ϕ. By Riemann-Hurwitz formula, we have $0 = K_Y + D = \phi^*(K_X + \sum_i X_i) + R = R$. By Zariski-Nagata purity theorem, ϕ is then unbranched on the algebraic torus $\mathbb{G}_m^d \subseteq X$. In particular,

$\phi^{-1}(\mathbb{G}_m^d) \subseteq Y$ is an étale cover of $\mathbb{G}_m^d$, so is itself an algebraic torus. Then Y can be viewed as the normalization of X in the field $K(\phi^{-1}(\mathbb{G}_m^d))/K(\mathbb{G}_m^d)$. Since each ϕ_i is toric, one can always construct a finite toric cover Y' of X such that the restriction on toric boundaries give the same morphisms $\{\phi_i | 1 \leq i \leq n\}$. Y' is normal, so $Y = Y'$, and hence Y is a toric variety with $\phi : Y \to X$ a toric morphism. $\square$

This finishes the proof of the necessity.

Chapter 4

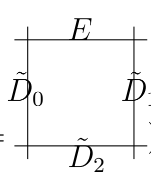
Examples

Example 4.1. *Let $Y = \mathbb{P}^d$ and $D = D_0 + D_1 + \cdots + D_d$, the union of coordinate planes. Then $K_Y + D = 0$. $\mathrm{Pic}(Y) \simeq \mathbb{Z}$, so $U \simeq \mathbb{Z}^d$ and $\mathrm{rank}(U) = d = \dim(Y)$. Meanwhile, (Y, D) is a toric pair.*

Example 4.2. *Let $d \geq 2$, $Y = \mathbb{P}^d$ and D be a nonsingular hypersurface of degree $d + 1$. Then $K_Y + D = 0$. $\mathrm{Pic}(Y) \simeq \mathbb{Z}$, so $U = 0$ and $\mathrm{rank}(U) = 0 < \dim(Y)$. Meanwhile, (Y, D) is a not toric pair (though Y is a toric variety).*

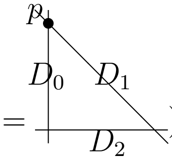
Example 4.3. *Let $Y = \mathbb{P}^2$ and $D = D_0 + D_1 + D_2$, the union of coordinate axes. Starting with this pair, we can perform two types of blow-ups:*

(1) Take $p \in \mathrm{Sing}(D) = \{[1:0:0], [0:1:0], [0:0:1]\}$. Let $\pi : \tilde{Y} \to Y$ be the blow-up along p, $\tilde{D} = \tilde{D}_0 + \tilde{D}_1 + \tilde{D}_2 + E$ be the total transform of D.

$(\tilde{Y} = \mathrm{Bl}_p(\mathbb{P}^2), \tilde{D} =$ 

$)$

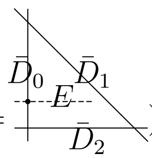
$\downarrow$

$(Y = \mathbb{P}^2, D =$ 

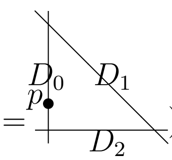
$)$

Then $K_{\tilde{Y}} = \pi^*K_Y + E = \pi^*(-(D_0+D_1+D_2)) + E = (-\tilde{D}_0 - E) + (-\tilde{D}_1 - E) + (-\tilde{D}_2) + E = -\tilde{D}_0 - \tilde{D}_1 - \tilde{D}_2 - E$. *So* $K_{\tilde{Y}} + \tilde{D} = 0$. $\mathrm{Pic}(\tilde{Y}) \simeq \mathrm{Pic}(Y) \oplus \mathbb{Z} \cdot E \simeq \mathbb{Z}^2$, *so* $\tilde{U} \simeq \mathbb{Z}^2$ *and* $\mathrm{rank}(\tilde{U}) = 2 = \dim(\tilde{Y})$. *Meanwhile,* $(\tilde{Y}, \tilde{D})$ *is a toric pair.*

(2) Take $p \in D \setminus \mathrm{Sing}(D)\}$. *Let* $\pi : \bar{Y} \to Y$ *be the blow-up along* p, $\bar{D} = \bar{D}_0 + \bar{D}_1 + \bar{D}_2$ *be the strict transform of* D.

$(\bar{Y} = \mathrm{Bl}_p(\mathbb{P}^2), \bar{D} =$  $)$

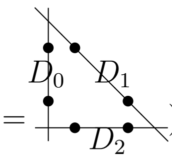
$$\downarrow$$

$(Y = \mathbb{P}^2, D =$  $)$

Then $K_{\bar{Y}} = \pi^*K_Y + E = \pi^*(-(D_0 + D_1 + D_2)) + E = (-\bar{D}_0 - E) + (-\bar{D}_1) + (-\bar{D}_2) + E = -\bar{D}_0 - \bar{D}_1 - \bar{D}_2$. *So* $K_{\bar{Y}} + \bar{D} = 0$. $\mathrm{Pic}(\bar{Y}) \simeq \mathrm{Pic}(Y) \oplus \mathbb{Z} \cdot E \simeq \mathbb{Z}^2$, *so* $\bar{U} \simeq \mathbb{Z}$ *and* $\mathrm{rank}(\bar{U}) = 1 < \dim(\bar{Y})$. *Meanwhile,* $(\bar{Y}, \bar{D})$ *is not a toric pair (because* $\bar{Y} \setminus \bar{D} = (Y \setminus D) \cup (E \setminus point) = \mathbb{T}^2 \cup \mathbb{A}^1$*).*

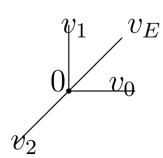
Case (1) is an example of the toric blow-up, while case (2) is an example of the toric model (see Definition 3.1). In general, a toric blow-up would keep U *while a non-toric blowup will decrease its rank (see the proof of Theorem 4). Meanwhile, a toric blow-up will turn a toric pair into another toric pair while a non-toric blow-up will turn a toric pair into a non-toric pair.*

Example 4.4. *In Example 4.3(2), though* $(\bar{Y}, \bar{D})$ *is not a toric pair,* $\bar{Y}$ *is still a toric surface as it is the blow-up of* $\mathbb{P}^2$ *at a single point, so isomorphic to* *the Hirzebruch surface $\mathbb{F}_1$. Now blow-up three pairs of points on $D \setminus \mathrm{Sing}(D)$, each of which lie on one coordinate axis.*

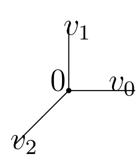
$(Y = \mathbb{P}^2, D =$  $)$

This is actually same as blowing-up six general points on $\mathbb{P}^2$: we can divide these general points into three pairs, use one line to connect each pair and then use the automorphisms of $\mathbb{P}^2$ to turn the three lines into the coordinate axes. So we will get a smooth cubic surface in $\mathbb{P}^3$, which is not a toric surface.

Example 4.5. *Let us exhibit explicitly the linear structure on dual simplicial complex Σ for simple examples. When (Y, D) is a toric pair, we are getting back to the complete fan corresponding to it. In particular, for Example 4.3(1), blowing-up at p corresponds to inserting a new ray.*

$\tilde{\Sigma} =$ 

$\uparrow$

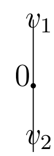
$\Sigma =$ 

This is no more than a naive case of [GHK11], Lemma 1.17, which states that a general toric blow-up corresponds to a refinement of the original Σ.

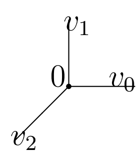
For Example 4.3(2), we have to calculate the group $\bar{U}$. By definition, $\bar{U}$ is the kernel of the homomorphism

$$\begin{array}{cccccc} \mathbb{Z}\cdot\bar{D}_0\oplus & \mathbb{Z}\cdot\bar{D}_1\oplus & \mathbb{Z}\cdot\bar{D}_2 & \to & \mathrm{Pic}(Y)\oplus & \mathbb{Z}\cdot E: \\ \bar{D}_0 & & & \mapsto & (1, & -1) \\ & \bar{D}_1 & & \mapsto & (1, & 0) \\ & & \bar{D}_2 & \mapsto & (1, & 0), \end{array}$$

so is generated by $\bar{D}_1 - \bar{D}_2$. We can write $\bar{\Sigma}$ in the following way:

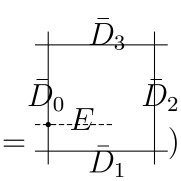
$\bar{\Sigma} =$ 

$$\uparrow$$

$\Sigma =$ 

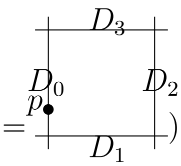
Roughly speaking, we are "collapsing" v_0.

Example 4.6. *It is worth noting that the proof of Proposition 17(1) does not depend on the assumption* $\mathrm{rank}(U) = d$. *So even when (Y, D) is not a toric pair,* $\mathrm{rank}(U) < d$, *we could still get a regular morphism $Y \to \mathrm{TV}(\Sigma)$. In Example 4.3(2), we are getting back to the standard fibration $\mathbb{F}_1 \to \mathbb{P}^1$. Let us consider another example: performing a non-toric blow-up to $\mathbb{P}^1 \times \mathbb{P}^1$.*

$(\bar{Y} = \mathrm{Bl}_p(\mathbb{P}^1 \times \mathbb{P}^1), \bar{D} =$ 

$)$

$$\downarrow$$

$(Y = \mathbb{P}^1 \times \mathbb{P}^1, D =$  $)$

$\bar{Y}$ is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at a single point, so isomorphic to the blowup of $\mathbb{P}^2$ at two points. This surface admits a morphism to $\mathrm{TV}(\bar{\Sigma}) \simeq \mathbb{P}^1$ simply by $\bar{Y} \xrightarrow{\pi} \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$.

Chapter 5

M$^{\text{c}}$Kernan's Conjecture

We have finished the proof of the Main Theorem. However, it is just the easiest case of M$^{\text{c}}$Kernan's conjecture. In this chapter I will present the full version, and list some attempts to it. But first, let us recall the standard definitions of singularities used in the log Minimal Model Program.

Definition 5.1. Let Y be a normal variety and $\Delta = \sum d_i \Delta_i$ a boundary (i.e., a $\mathbb{Q}$-divisor of which all coefficients lie between zero and one, $0 \leq d_i \leq 1$), such that $K_Y + \Delta$ is $\mathbb{Q}$-Cartier. Let $f : X \to Y$ be a proper birational morphism from a normal variety X. Then we can write

$$K_X = f^*(K_Y + \Delta) + \sum a(E, \Delta) E,$$

where the sum runs over all the distinct prime divisors $E \subset X$, and $a(E, \Delta) \in \mathbb{Q}$. We call $a(E, \Delta)$ the discrepancy of E with respect to (Y, Δ), and define $\operatorname{discrep}(Y, \Delta) = \inf_E \{a(E, \Delta) | E \text{ is exceptional over } Y\}$. We say that (Y, Δ) is

$$\begin{cases} \text{terminal} \\ \text{canonical} \\ \text{klt (= Kawamata log terminal)} \\ \text{plt (= purely log terminal)} \\ \text{lc (= log canonical)} \end{cases} \text{if } \operatorname{discrep}(Y, \Delta) \begin{cases} > 0 \\ \geq 0 \\ > -1, \text{ and } \lfloor \Delta \rfloor = 0 \\ > -1 \\ \geq -1. \end{cases}$$

A more delicate notion is the dlt (= divisorial log terminal) singularity. It has two equivalent definitions, though between which the equivalence is non-trivial.

Definition 5.2. Let Y be a normal variety and $\Delta = \sum d_i \Delta_i$ a boundary, such that $K_Y + \Delta$ is $\mathbb{Q}$-Cartier. We say that (Y, Δ) is dlt if

(1) There exists a log resolution $f : X \to Y$ such that $a(E, \Delta) > -1$ for every f-exceptional divisor E.

Or equivalently ([Sz94]),

(2) There exists a closed subset $Z \subset Y$ such that: (i) $Y \setminus Z$ is smooth and $\Delta|_{Y \setminus Z}$ is simple normal crossing; (ii) If $f : X \to Y$ is birational and $E \subset X$ is an irreducible divisor with center $\operatorname{center}_Y E \subset Z$, then $a(E, \Delta) > -1$.

An easy relation between these notions are klt $\Longrightarrow$ plt $\Longrightarrow$dlt $\Longrightarrow$ lc. One key feature of dlt singularities is they admit adjunction formula as following. Let (Y, Δ) be a dlt pair. Write $D = \lfloor \Delta \rfloor$ and let $D = \sum_{1 \le i \le n} D_i$ be the irreducible decomposition of D. Call subvariety $W \subseteq Y$ a lc center for the pair (Y, Δ) with $\operatorname{codim}_Y W = k$ if and only if W is an irreducible component of D_I with $I \subseteq \{1, \dots, n\}$ and $|I| = k$. Then for any lc center W, there exists a boundary Δ_W ("different") of W such that (W, Δ_W) is dlt, and $K_W + \Delta_W = (K_Y + \Delta)|_W$. However, in general, even if $D = \Delta$ is integral, Δ_W is not necessarily integral–which is different from the case that Y is smooth and $D = \Delta$ is a simple normal crossing divisor.

It is worth mentioning that the above is not the general definition of lc centers. In general, for a lc pair (Y, Δ), an irreducible subvariety $Z \subseteq Y$ is called a lc center if there is a birational morphism $f : X \to Y$ and a divisor $E \subset X$ such that $a(E, \Delta) = -1$ and $f(E) = Z$. Under this setting, a lc pair (Y, Δ) is dlt if and only if none of its lc centers is contained in $Y \setminus \operatorname{snc}(Y, \Delta)$.

We are now ready to state M^{c}Kernan's Conjecture.

Definition 5.3. Let Y be an irreducible variety of dimension d, Δ be a boundary. Let n be the sum of the coefficients of Δ. The components of Δ generate a subgroup of the Weil divisors modulo algebraic equivalence, of which the rank is denoted by r and called the rank of Δ. The absolute rank R of Y is the rank of the group of all Weil divisors modulo algebraic equivalence. Moreover, the charge (or complexity) $c(Y, \Delta)$ of the pair is defined as $r+d-n$, and the absolute charge $C(Y, \Delta)$ is defined as $R+d-n$ (so $c(Y, \Delta) \leq C(Y, \Delta)$).

Conjecture 1 (M^{c}Kernan)**.** *Let Y be a proper variety of dimension n and let Δ be a boundary. Assume that (Y, Δ) is log canonical and $-(K_Y + \Delta)$ is nef. Then*

(1) $c(Y, \Delta) \geq 0$.

(2) If $C(Y, \Delta) < 2$ then Y is geometrically rational (i.e., rational over the algebraic closure of the base field).

(3) If $c(Y, \Delta) < 1$ then there is a divisor D such that the pair (Y, D) is toric. Moreover, $\lfloor \Delta \rfloor \subseteq D$ and $D - S$ is linearly equivalent to a divisor with support in Δ, where S is either empty or an irreducible divisor.

Remark 5.1. (i) Let Y be an elliptic curve and let Δ be empty. Then $C(Y,\Delta) = 2$ and $c(Y,\Delta) = 0$. Y is definitely not geometrically rational. This shows we can not replace the absolute charge by charge in part (2).

(ii) Let $Y = \mathbb{F}_n$ be the Hirzebruch surface, $\Delta = 2E_\infty + \sum F_i$ with E_∞ the negative section and F_i fibers of the natural projection $\mathbb{F}_n \to \mathbb{P}^1$. Then $K_Y + \Delta = 0$, $c(Y,\Delta) \leq 0$ but the pair (Y,Δ) is non-toric. This shows we can not loose the assumption that Δ is a boundary. Moreover, by contracting E_∞ we can make the image of Δ become a boundary, but the image of (Y,Δ) is still not a toric pair. This shows we can not loose the assumption that (Y,Δ) is log canonical.

One of the motivations for the conjecture is to answer the following

MOTIVATING QUESTION: Why are toric varieties so ubiquitous?

The inspiration of this conjecture arises from the theory of complements. It is also a generalization of an earlier conjecture by Shokurov. Unfortunately, a counterexample to both conjectures was recently found by Karzhemanov ([Ka13]) at dimension $d \geq 3$.

Example 5.1. *([Ka13]) Let $[x_0 : x_1 : x_2]$ be projective coordinates on $\mathbb{P}^2$ and $G := \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ act on $\mathbb{P}^2$ by $[x_0 : x_1 : x_2] \mapsto [\pm x_0 : x_1 : \pm x_2]$. Pick a G-invariant point on $\mathbb{P}^2$, denoted by p. Take $W = \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is the unique indecomposable vector bundle (up to isomorphism) on $\mathbb{P}^2$ with Chern class $(c_0, c_1, c_2) = (2, 0, 1)$ and admitting a splitting $0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{I}_p \to 0$. Then $g^*(E) = E$ for any $g \in G$. Hence the G-action lifts from $\mathbb{P}^2$ to a regular* *action on W. Let $Y = W/G$ be the quotient. One could then find a delicately chosen boundary divisor Δ such that (Y, Δ) is log canonical, $K_Y + \Delta \equiv 0$, and $C(Y, \Delta) = 1/2$. However, Y is not a toric variety.*

The example shows that the original statement of the conjecture has to be modified a little bit, and it is still a open problem to look for the full precise statement.

Here are some cases we already know:

(i) $d = 2$. The surface case was proved by Shokurov ([Sh00]).

(ii) $d = 3$, (Y, Δ) is plt and $K_Y + \Delta \equiv 0$. Under this condition, the conjecture was partly proved by Prokhorov ([Pr01]).

(iii) $D = \Delta$ is integral, Y is $\mathbb{Q}$-factorial and $K_Y + D \equiv 0$. Under this condition, it is known that part (1) holds and if either $C(Y, D) = 0$ or Y is rationally connected and $C(Y, D) = 1$, then part (2) holds. This was proved by Karzhemanov and Prokhorov ([Ka13]).

(iv) $D = \Delta$ is integral, Y is projective and $\mathbb{Q}$-factorial and the characteristic is zero. M$^{\text{c}}$Kernan gave an unpublished sketch of a proof, assuming that the Minimal Model Program holds.

(v) $D = \Delta$ is integral, Y is a smooth projective variety over $\mathbb{C}$ and $K_Y + D \equiv 0$. Then parts (1) and (3) of the conjecture are equivalent to our Main Theorem.

Remark 5.2. The key Proposition 4 by Kollár ([Kol11], Theorem 10) works for any $\mathbb{Q}$-factorial, dlt pair with integral boundary. In other words, the dual simplicial complex always has a nice topology as in Proposition 5. But it is unknown to me under what assumptions we could further endow a linear structure to make it a complete fan and relate Y to the corresponding toric variety.

Remark 5.3. By a theorem of Hacon ([KK10]), every log canonical pair admits a $\mathbb{Q}$-factorial, crepant (meaning that $K_X + \Delta_X = f^*(K_Y + \Delta_Y)$), dlt model. So in philosophy, we should be able to pass the information about the dlt case to the more general log canonical case.

Bibliography

[Ab06] M. Abouzaid, *Homogeneous coordinate rings and mirror symmetry for toric varieties*, Geom. Topol. 10 (2006), 1097-1157.

[AKMW02] D. Abramovich, K. Karu, K. Matsuki, J. Włodarczyk, *Torification and factorization of birational maps*, J. Amer. Math. Soc. 15 (2002), no. 3, 531572.

[Au07] D. Auroux, *Mirror symmetry and T-duality in the complement of an anticanonical divisor*, J. GÖkova Geom. Topol. GGT 1 (2007), 51-91.

[BCHM10] C. Birkar, P. Cascini, C. Hacon, J. M^{c}Kernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. 23 (2010), no. 2, 405-468.

[CLS11] D. Cox, J. Little, H. Schenck, *Toric varieties*, Graduate Studies in Mathematics, 124. American Mathematical Society, Providence, RI, 2011.

[Co07] A. Corti (eds.), *Flips for 3-folds and 4-folds*, Oxford Lecture Series in Mathematics and its Applications, 35. Oxford University Press, Oxford, 2007.

[FM83] R. Friedman, D. Morrison (eds.), *The birational geometry of degenerations*, Progr. Math., 29, Birkhäuser, Boston, Mass., 1983.

[Fu93] W. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, 131. Princeton University Press, Princeton, NJ, 1993.

[GHK11] M. Gross, P. Hacking, S. Keel, *Mirror symmetry for log Calabi-Yau surfaces I*, arXiv:1106.4977.

[GMN09] D. Gaiotto, G. Moore, A. Neitzke, *Wall-crossing, Hitchin systems, and the WKB approximation*, arXiv:0907.3987.

[GS11a] M. Gross, B. Siebert, *From real affine geometry to complex geometry*, Ann. of Math. (2) 174 (2011), no. 3, 1301-1428.

[GS11b] M. Gross, B. Siebert, *Logarithmic Gromov-Witten invariants*, arXiv:1102.4322.

[Ha77] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, 52. Springer-Verlag, New York-Heidelberg, 1977.

[HK00] Y. Hu, S. Keel, *Mori dream spaces and GIT*, Michigan Math. J. 48 (2000), 331-348.

[HM^{c}10] C. Hacon, J. M^{c}Kernan, *Existence of minimal models for varieties of log general type. II*, J. Amer. Math. Soc. 23 (2010), no. 2, 469-490.

[HMo98] J. Harris, I. Morrison, *Moduli of curves*, Graduate Texts in Mathematics, 187. Springer-Verlag, New York, 1998.

[Ka13] I. Karzhemanov, *On characterization of toric varieties*, arXiv:1306.4131v2.

[KK10] J. Kollár, S. Kovács, *Log canonical singularities are Du Bois*, J. Amer. Math. Soc. 23 (2010), no. 3, 791813.

[KM98] J. Kollár, S. Mori, *Birational geometry of algebraic varieties*, With the collaboration of C. H. Clemens and A. Corti, Cambridge Tracts in Mathematics, 134. Cambridge University Press, Cambridge, 1998.

[Kol96] J. Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, Bd. 32. Springer-Verlag, Berlin, 1996.

[Kol11] J. Kollár, *Sources of log canonical centers*, arXiv:1107.2863v3.

[KS06] M. Kontsevich, Y. Soibelman, *Affine structures and non-Archimedean analytic spaces*, The unity of mathematics, 321-385, Progr. Math., 244, Birkhäuser Boston, Boston, MA, 2006.

[Mo92] D. Morrison, *Compactifications of moduli spaces inspired by mirror symmetry*, Journées de Géomtrie Algébrique d'Orsay (Orsay, 1992), Astérisque No. 218 (1993), 243271.

[Pr01] Y. Prokhorov, *On a conjecture of Shokurov: characterization of toric varieties*, Tohoku Math. J. (2) 53 (2001), no. 4, 581592.

[Sh00] V. Shokurov, *Complements on surfaces*, J. Math. Sci. (New York) 102 (2000), no. 2, 38763932.

[SYZ96] A. Strominger, S.-T. Yau, E. Zaslow, *Mirror symmetry is T-duality*, Nuclear Phys. B 479 (1996), no. 1-2, 243-259.

[Sz94] E. Szabó, *Divisorial log terminal singularities*, J. Math. Sci. Univ. Tokyo 1 (1994), no. 3, 631639.

[Ya13] Y. Yao, *A criterion for toric varieties*, preprint.

[YYY13] Y.-H. Yang, Y. Yao, Y. Ye, *(Quasi-)Poisson enveloping algebras*, Acta Math. Sin. (Engl. Ser.) 29 (2013), no. 1, 105118.

[YYZ07] Y. Yao, Y. Ye, P. Zhang, *Quiver Poisson algebras*, J. Algebra 312 (2007), no. 2, 570-589.

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